Local Minima and Maxima of a Functional: Determining the nature of Solutions to the Euler-Lagrange Equations

Matthew Neeley

March 11, 2005

Suppose we have a functional $S[x(t)]$ that maps a function $x : [a, b] \to \mathbb{R}$ on a given interval to a number. Using the Euler-Lagrange equation, we can find extremal or "stationary" paths of this functional, that is, paths for which

$$\delta S[x_0(t)] = 0$$

However, this does not of itself tell us whether the extremal path is a local minimum, maximum or saddle point of the functional, just as in conventional calculus, finding a point at which the derivative of a function vanishes does not tell us the nature of the extremum. In the case of conventional calculus, we compute the second derivative, and its sign (assuming it does not vanish) tells us whether the point in question is a local minimum or maximum. If the second derivative vanishes, we could look at higher derivatives.

We consider the following functional on the interval $[0, T]$:

$$S[x(t)] = \int_0^T dt \left(\dot{x}^2 + x^2\right)$$

_We wish to find the path between points $(0, 0)$ and $(T, \sinh T)$that extremizes this functional. Applying the Euler-Lagrange equation gives

$$\ddot{x} + x = 0 \tag{1}$$

which has the solution $x = \sinh t$ that clearly satisfies the boundary conditions. We now wish to determine whether this function is a local minimum or maximum of the functional S. If we vary x by adding δx which is assumed to vanish at $t = 0$ and $t = T$, then the variation in S is:

$$\delta S = S[x + \delta x] - S[x] = \int_0^T dt \left(\left(\dot{x} + \dot{\delta x}\right)^2 + (x + \delta x)^2 - \dot{x}^2 - x^2\right)$$

$$= \int_0^T dt \left(2\dot{x}\delta\dot{x} + \delta\dot{x}^2 + 2x\delta x + \delta x^2\right)$$

The first term can be integrated by parts to give a boundary term $2\dot{x}\delta x$ which vanishes since the variation is zero at the endpoints. This leaves

$$= \int_0^T dt \left(-2\ddot{x}\delta x + \delta\dot{x}^2 + 2x\delta x + \delta x^2\right)$$

Now, by the Euler-Lagrange equation (1) the first and third terms will cancel, leaving

$$\int_0^T dt \left(\delta\dot{x}^2 + \delta x^2\right) \geq 0$$

which is clearly greater than zero since the integrand is positive definite. Equality follows only for $\delta x = 0$ so that indeed, the path $x(t)$ found from the Euler-Lagrange equations is a local minimum. In fact, in this case it is a *global* minimum, since there is only one stationary path, and any other path yield a larger value of S as we have shown.